From Bonham-Carter, Graeme F., Geographic Information Systems for Geoscientists, Modelling with GIS, Chapter 9, Fuzzy Logic section with related tables and figures.

FUZZY LOGIC METHOD

In classical set theory, the membership of a set is defined as true or false, 1 or 0. Membership of a fuzzy set, however, is expressed on a continuous scale from 1 (full membership) to 0 (full non-membership). Thus individual measurements of arsenic (As) in lake sediment might be defined according to their degree of membership in the set called "Arsenic anomaly". Very high values of As are definitely anomalous, with a fuzzy membership of 1; very low values at or below background have a fuzzy membership of zero; between these extremes a range of possible membership values exist. Such a membership function might be expressed analytically as

$$\mu(x) = \begin{cases} 0 & x < 50 \\ \dfrac{x - 50}{200} & 50 < x < 250 \\ 1 & x > 250 \end{cases} \qquad \textbf{(9-3)}$$

where x is the As concentration value in ppm and μ(x) is the fuzzy membership function. Every value of x is associated with a value of μ(x), and the ordered pairs [x, μ(x)] are known collectively as a fuzzy set. The shape of the function need not be linear, as in Equation 9-3 and shown in Figure 9-5, it can take on any analytical or arbitrary shape appropriate to die problem at hand. Fuzzy membership functions can also be expressed as lists or tables of numbers. Thus in the As case, the discrete representation of the fuzzy membership function shown in Table 9-7 (for class intervals of 50 ppm), is equivalent to the analytical expression in Equation 9-3.

Now suppose that As has been mapped, with 50 ppm class intervals, then the fuzzy membership function can be treated as an attribute table of an arsenic map, as shown in Table 9-8.

The classes of any map can be associated with fuzzy membership values in an attribute table. The level of measurement of the mapped variable can be categorical, ordinal or interval.

Fuzzy membership values must lie in the range (0, 1), but there are no practical constraints on the choice of fuzzy membership values. Values are simply chosen to reflect the degree of membership of a set, based on subjective judgment. Values need not increase or decrease monotonically with class number, as in the case of As above.

The presence of the various states or classes of a map might be expressed in terms of fuzzy memberships of different sets, possibly storing them as several fields in the map attribute table. Thus the As values on a map might be considered in terms of their fuzzy membership of a set'*"favourable indicator for gold deposits"*, or a second set *"suitable for drilling water wells"*. The membership functions for these two sets would look very different, one reflecting the importance of As as a pathfinder element for gold deposits, the other reflecting the undesirability of drilling a water well in rocks with elevated levels of As.

Not only can a single map have more than one fuzzy membership function, but also several different maps can have membership values for the same proposition or hypothesis. Suppose that the spatial objects (polygons, pixels) on a map, are evaluated according to the proposition *"favourable location for gold exploration"* then any of the maps to be used as evidence in support of this proposition can he assigned fuzzy membership functions. Table 9-5 shows a series of fuzzy membership functions for the maps used to select a landfill. Ale membership values were chosen arbitrarily (like the index overlay scores) based on subjective judgment about the relative importance of the maps and their various states. The fuzzy membership values are in the field labelled "Fuzzy". Table 9-6 also shows fuzzy membership functions for the mineral potential maps. Note that the fuzzy memberships assigned to categorical maps (such as the geological map or the zoning map in the landfill study) do not increase or decrease monotonically with class number, but are assigned values in the range (0,1) that reflect, subjectively, the importance of individual map units. Thus limestone is assigned a value of 0.1 (highly unfavourable for a landfill), whereas a shale is assigned a very favourable value (0.9).

Note that the fuzzy membership values must reflect the relative importance of each map, as well as the relative importance of each class of a single map. The fuzzy memberships are similar to the combined effect of the class scores and the map weights of the index overlay method.

Table 9-6. Attribute tables for mineral potential study, showing scores for class weighting and fuzzy membership values. Only 4 out of the 1 0 tables used in the study are shown. Tables have been assigned the same names as their associated maps.

A. Geology (GEOL)			
Class	**Score**	**Fuzzy**	**Legend**
0	-1	0.0	'outside'
1	9	0.8	'Goldenville'
2	7	0.7	'Halifax'
3	-1	0.1	'Granite'
B. Lake Sediment Antimony (LSSB)			
0	1	0.1	'no data'
1	8	0.8	'0.9-1.3 ppm'
2	7	0.8	'0.8-0.9'
3	6	0.6	'0.6-0.8'
4	5	0.4	'0.5-0.6'
5	4	0.3	'0.4-0.5'
6	2	0.2	'0.3-0.4'
7	2	0.2	'0.2-0.3'
8	1	0.1	'<0.2'
C. Balsam Fir Gold (BIOAU)			
0	0	0.0	'no data'
1	9	0.9	'24-136 ppb'
2	8	0.8	'16-24'
3	8	0.6	'12-16'
4	7	0.4	'10-12'
5	6	0.3	'8-10'
6	5	0.2	7-8'
7	4	0.2	'6-7'
8	2	0.2	'3-6'
9	1	0.1	'<3'

D. Anticline distance (ANTI)			
Class	**Score**	**Fuzzy**	**Legend**
0	0	0.1	'> 6 km'
1	9	0.9	'<0.25'
2	9	0.9	'0.25-0.5'
3	9	0.9	'0.5-0.75'
4	9	0.9	'0.75-1.0'
5	8	0.8	'1.0'1.25'
6	8	0.8	'1.25-1.5'
7	8	0.8	'1.5-1.75'
8	8	0.8	'1.75-2.0'
9	7	0.7	'2.0-2.25'
10	7	0.7	'2.25-2.5'
11	7	0.7	'2.5-2.75'
12	6	0.6	'2.75-3.0'
13	6	0.5	'3.0-3.25'
14	6	0.5	'3.25-3.5'
15	5	0.5	'3.5-3.75'
16	5	0.4	'3.75-4.0'
17	4	0.4	'4.0-4.25'
18	4	0.4	'4.25-4.5'
19	3	0.3	'4.5-4.75'
20	3	0.3	'4.75-5.0'
21	2	0.3	'5.0-5.25'
22	2	0.3	'5.25-5.5'
23	1	0.2	'5.5-5.75'
24	1	0.2	'5.75-6.0'

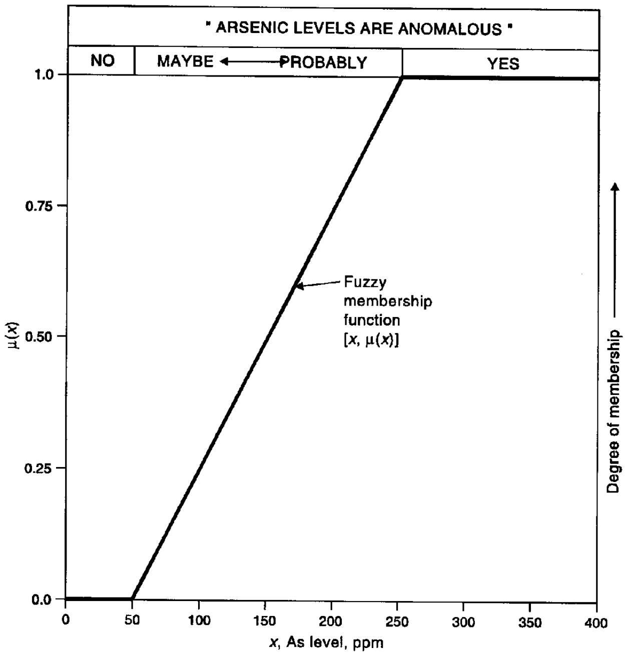


FIG. 9-5. A graph showing fuzzy membership of the set of observations for which *"arsenic levels are anomalous"*. Fuzzy membership can, in some cases, he expressed as an analytical function, not necessarily linear as shown here, in other cases membership is defined more readily as a table.

Table 9-7. Fuzzy membership function for As expressed as the ordered pairs *[x,* μ *(x)l,* and organized in a table.

x	μ*(x)*
300	1
250	1
200	0.75
150	0.5
100	0.25
50	0
0	0

Table 9-8. Attribute table for a map of As, with the fuzzy membership values shown as one field.

Map Class	Fuzzy Membership	Legend Entry
1	1.00	'>275 ppm As'
2	1.00	'225 - 275'
3	0.75	'175 - 225'
4	0.50	'125 - 175'
5	0.25	'75 - 125'
6	0.00	'25 - 75'
7	0.00	'< 25'

Table 9-5. Attribute tables for the 10 maps used for landfill site selection.

A. Overburden thickness (OVERTHIK)

Class	Fuzzy	Legend
1	0.1	“1 m”
2	0.3	“2 m”
3	0.9	“3 m”
4	0.9	“4 m”
5	0.9	“5 m”
6	0.9	“6 m”

C. Surface slope (SLOPE)

Class	Fuzzy	Legend
1	0.9	“low”
2	0.9	
3	0.7	
4	0.5	“medium”
5	0.3	
6	0.1	
7	0.1	
8	0.1	“steep”

G. 100-year flood zone (FLOOD)

Class	Fuzzy	Legend
1	0.1	“100 yr”
2	0.9	“> 100"

H. Suitability for farming (SUITAB)

Class	Fuzzy	Legend
1	0.1	“good”
2	0.4	“fair”
3	0.9	“poor”

I. Distance from major road (ROADBUF)

Class	Fuzzy	Legend
1	0.6	“<1 km”
2	0.9	“<2"
3	0.8	“<3"
4	0.7	“<4"
5	0.5	“<5"
6	0.3	“<6"
7	0.1	“<7"
8	0.1	“<8"

B. Permeability (PERMEAB)

Class	Fuzzy	Legend
1	0.9	“low”
2	0.6	“med”
3	0.2	“high”

D. Geology (GEOLOGY)

Class	Fuzzy	Legend
1	0.8	“granite”
2	0.5	“sandstone”
3	0.9	“shale”
4	0.1	“limestone”
5	0.2	“conglomerate”

E. Zoning map (ZONING)

Class	Fuzzy	Legend
0	0.1	“city”
1	0.3	“industrial”
2	0.8	“agricult A”
3	0.7	“agricult B”

F. Distance from city limits (MUNIBUF)

Class	Fuzzy	Legend
0	0.0	“0 km”
1	0.6	“<1 km”
2	0.8	“<2"
3	0.9	“<3"
4	0.7	“<4"
5	0.5	“<5"
6	0.3	“<6"
7	0.1	“<7"
8	0.1	“<8"
9	0.1	“<9"
10	0.1	“<10"
11	0.1	“>=10"

J. Ecologically sensitive (ECOL)

Class	Fuzzy	Legend
1	0.1	“sensitive”
2	0.9	“insensitive”

Combining Fuzzy Membership Functions

Given two or more maps with fuzzy membership functions for the same set, a variety of operators can be employed to combine the membership values together. The book by Zimmermann (1985), for example, discusses a variety of combination rules. An et al. (1991) discuss five operators that were found to be useful for combining exploration datasets, namely the fuzzy AND, fuzzy OR, fuzzy algebraic product, fuzzy algebraic sum and fuzzy gamma operator. These operators are briefly reviewed here.

Fuzzy AND

This is equivalent to a Boolean AND (logical intersection) operation on classical set values of (1,0). It is defined as

$$\mu_{combination} = MIN(\mu_A, \mu_B, \mu_C, ...) \qquad (9\text{-}4)$$

where μ_A is the membership value for map A at a particular location, μ_B, is the value for map B, and so on. Of course, the fuzzy memberships must all be with respect to the same proposition. Suppose that at some location the membership value for map A is 0.75 and for map B is 0.5, then the membership for the combination using fuzzy AND is 0.5. It can readily be seen that the effect of this rule is to make the output map be controlled by the smallest fuzzy membership value occurring at each location. Like the Boolean AND, fuzzy AND results in a conservative estimate of set membership, with a tendency to produce small values. The AND operation is appropriate where two or more pieces of evidence for a hypothesis must be present together for the hypothesis to be true.

Fuzzy OR

On the other hand, the fuzzy OR is the like the Boolean OR (logical union) in that the output membership values are controlled by the maximum values of any of the input maps, for any particular location. The fuzzy OR is defined as

$$\mu_{combination} = MAX(\mu_A, \mu_B, \mu_C, ...) \qquad (9\text{-}5)$$

Using this operator, the combined membership value at a location (=suitability for landfill etc) is limited only by the most suitable of the evidence maps. This is not a particularly desirable operator for the landfill case, but might in some circumstances be reasonable for mineral potential mapping, where favourable indicators of mineralization are rare and the presence of any positive evidence may be sufficient to suggest favourability. Note that in using either the fuzzy AND or fuzzy OR, a fuzzy membership of a single piece of evidence controls the output value. On the other hand, the following operators combine the effects of two or more pieces of evidence in a "blended" result, so that each data source has some effect on the output.

Fuzzy Algebraic Product

Here, the combined membership function is defined as

$$\mu_{combination} = \prod_{i=1}^{n} \mu_i \tag{9-6}$$

where μ_i is the fuzzy membership function for the i-th map, and i=1,2,... n maps are to be combined. The combined fuzzy membership values tend to be very small with this operator, due to the effect of multiplying several numbers less than 1. The output is always smaller than, or equal to, the smallest contributing membership value, and is therefore "decreasive". For example, the algebraic product of (0.75, 0.5) is 0.375. Nevertheless, all the contributing membership values have an effect on the result, unlike the fuzzy AND, or fuzzy OR operators.

Fuzzy Algebraic Sum

This operator is complementary to the fuzzy algebraic product, being defined as

$$\mu_{combination} = 1 - \prod_{i=1}^{n} (1 - \mu_i) \tag{9-7}$$

The result is always larger (or equal to) the largest contributing fuzzy membership value. The effect is therefore "increasive". Two pieces of evidence that both favour a hypothesis reinforce one another and the combined evidence is more supportive than either piece of evidence taken individually. For example, the fuzzy algebraic sum of (0.75, 0.5) is 1-(1-0.75)*(1-0.5), which equals 0.875. The increasive effect of combining several favourable pieces of evidence is automatically limited by the maximum value of 1.0, which can never be exceeded. Note that whereas the fuzzy algebraic product is an algebraic product, the fuzzy algebraic sum is not an algebraic summation.

Gamma Operation

This is defined in terms of the fuzzy algebraic product and the fuzzy algebraic sum by = *(Fuzzy algebra i c sum)*

$$\mu_{combination} = (\text{Fuzzy algebraic sum})^{\gamma} * (\text{Fuzzy algebraic product})^{1-\gamma} \tag{9-8}$$

where γ is a parameter chosen in the range (0, 1), Zimmermann and Zysno (1980). When γ is 1, the combination is the same as the fuzzy algebraic sum; and when γ is 0, the combination equals the fuzzy algebraic product. Judicious choice of γ produces output values that ensure a flexible compromise between the "increasive" tendencies of the fuzzy algebraic sum and the "decreasive" effects of the fuzzy algebraic product. For example, if γ= 0.7, then the combination of (0.75, 0.5) is $0.875^{0.7} * 0.375^{0.3}$ = 0.679, a result that lies between 0.75 and 0.5. On the other hand, if γ=0.95, then the combination is 0.839, a mildly increasive result. If γ=0. 1, then the combination is 0.408, a result that is less than the average of the 2 input function values, and is therefore decreasive. The effect of choosing different values of γ are shown in Figure 9-6. Note that although the same tendencies occur, the actual values of γ for which the combined membership function becomes increasive or decreasive vary with the input membership values. An et al. (1991) used a value of γ=0.975 to combine geophysical and geological datasets in their study of iron and base metal deposits in Manitoba, presumably because the increasive effects of larger values best seemed to reflect the subjective decision-making of typical exploration geologists.

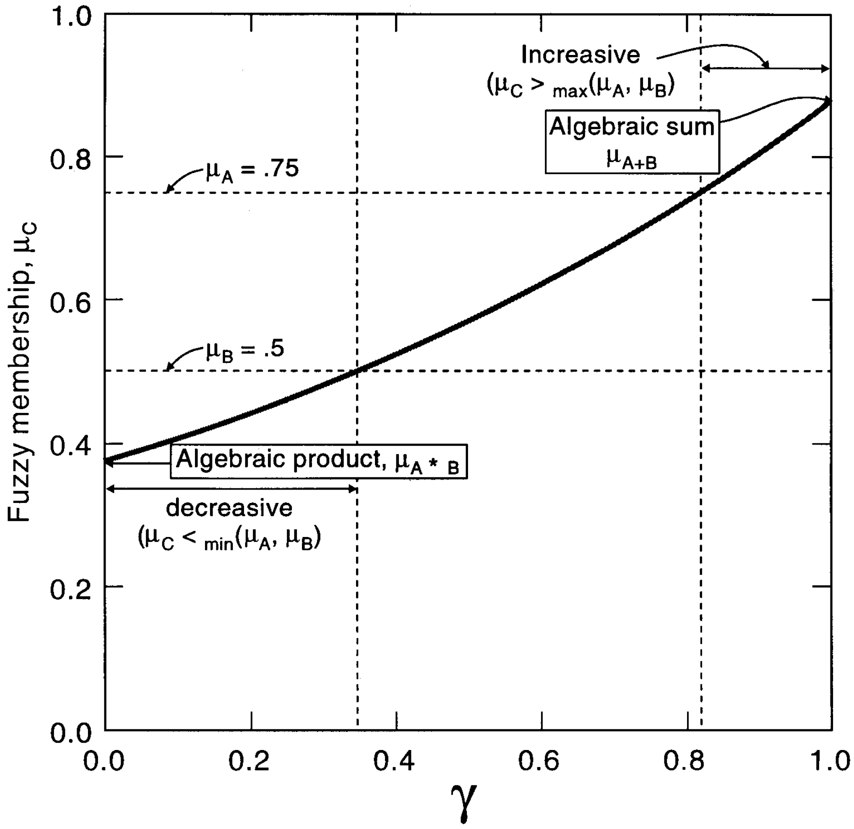


FIG. 9-6. A graph of fuzzy membership, μ_c, obtained by combining two fuzzy memberships, μ_A and μ_B, versus γ. This shows the effect of variations in γ for the case of combining two values, μ_A=0.75 and μ_B=0.5. When, γ=-0, the combination equals the fuzzy algebraic product; when γ=1, the combination equals the fuzzy algebraic sum.. When 0.8 < γ< 1, the combination is larger than the largest input membership value (in this case 0.75), and the effect is therefore "increasive". When 0 <γ< 0.35, the combination is smaller than the smallest input membership value (0.5 in this case), and the effect is therefore "decreasive". When 0.35<γ< 0.8, the combination is neither increasive nor decreasive, but lies within the range of the input membership values. The limits 0.8 and 0.35 are data dependent.

Returning to the internal modelling procedure, with the landfill case, the following steps can used to combine the 10 maps with the fuzzy gamma operation.

```
'Pseudocode for fuzzy combination of datasets for landfill site (see Table 9-5)
'Set value of Gamma

        gamma = 0.95

' At current location, lookup fuzzy membership values for each input map

        c1 = OVERTHIK
        c2 = PERMEAB
        c3 = SLOPE
        c4 = GEOLOGY
        c5 = FLOOD
        c6 = ZONING
        c7 = SUITAB
        c8 = MINIBUF
        c9 = ROADBUF
        c10 = ECOLOG

'Calculate the fuzzy algebraic product and fuzzy algebraic sum

        product = c1 * c2 * c3 * c4 * c5 * c6 * c7 * c8* c9 * c10
        sum = 1 - ((1 - c1) * (1 - c2) * (1 - c3) * (1 - c4) * (1 - c5) * (1 - c6) * (1 - c7) * (1 - c8) - (1 - c9) *
      (1 - c10)

'Apply gamma operator

        result = (sum ^ gamma) * (product ^ (1 - gamma))
```

Notice that for each of the 10 input maps, the 'FUZZY' column is the field in the corresponding map attribute table where the fuzzy membership functions are stored, see Table 9-5.

The output map, after classification with a table of breakpoints called 'FUZTAB', show areas ranked according the combined fuzzy membership, see Figure 9-2D.

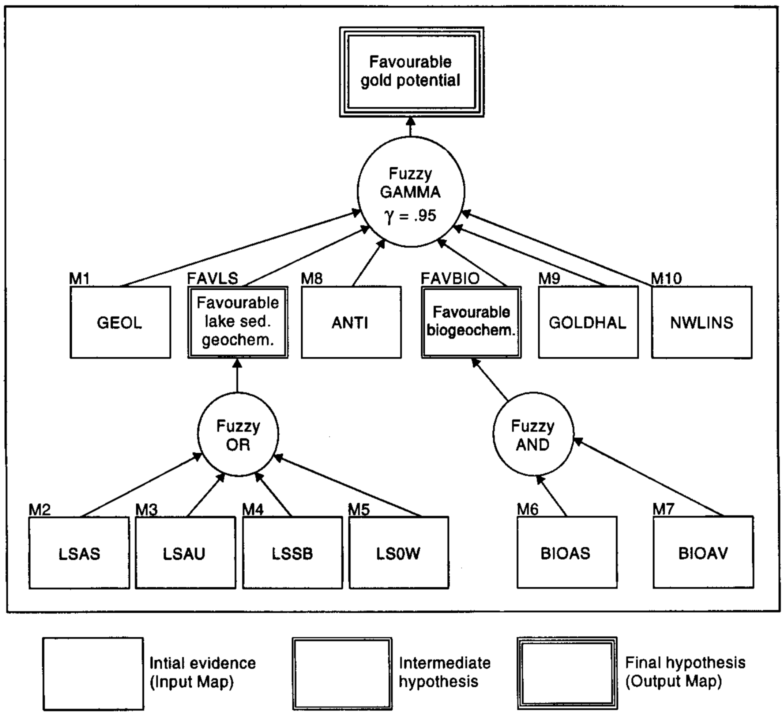
The procedure for the mineral potential case is similar, except for two features. First, the value of gamma is specified as keyboard input allowing different values to be selected at run time. Second, the four lake sediment maps are combined using fuzzy OR, and the two biogeochemical maps are also combined with fuzzy OR. This means the combined effect of the lake sediment geochemical evidence, will take on the maximum fuzzy membership of the four contributing maps. An anomalous value from any one of the maps is therefore sufficient to give this factor a large fuzzy score. The effect is the same for the biogeochemical combination. Finally, the gamma operator is used, as before, for the final combination step. The resulting map is shown in Figure 9-4D. Superficially it looks similar to the index overlay, but careful comparison shows some important differences.

```
'Pseudocode for fuzzy combination for mineral potential
'This procedure is shown graphically as an inference net in Figure 9-7
'Set gamma value
        gamma = 0.95
'At current location, get fuzzy membership values for each map
        m1 = GEOL
        m2 = LSAS
        m3 = LSAU
        m4 = LSSB
        m5 = LS0W
        m6 = BIOAS
        m7 = BIOAU
        m8 = ANTI
        m9 = GOLDHAL
        m10 = NWLINS
'Apply fuzzy OR to lake sediment maps
'Favourable lake sed geochem is an intermediate hypothesis
        favls = MAX(m2,m3,m4,m5)
'Favourable biogechem is an intermediate hypothesis
        favbio = MIN(m6,m7)
'Calculate fuzzy product, sum and gamma
'Favourable location for gold deposits is a final hypothesis
        fprod = m1 * favls * favbio * m8 * m9 * m10
        fsum = 1 - (( 1 - m1 ) * (1 - favls) *(1 - favbio)*(1 - m8) * (1 - m9)*(1 - m10))
        favloc = fsum ^ gamma * fprod ^ (1 - gamma)
```

Comments on the Fuzzy Logic Method

In practice, it may be desirable to use a variety of different fuzzy operators in the same problem, as shown for the mineral potential example. In particular, fuzzy AND and fuzzy OR can be more appropriate than fuzzy gamma in some situations, but not in others. For example, suppose that two input maps represent evidence for a proposition that requires that the evidence occur jointly. To take a slightly contrived example, consider a map of sulphur content and a map of zinc content from lithological samples. The combination is highly suggestive evidence for the presence of zinc sulphide (sphalerite), an important mineral in many zinc deposits. Ignoring the obvious problems of concentration units and other factors for the sake of simplicity, we can deduce that because the joint presence of the two elements is needed, the importance of the evidence is limited by the lesser abundance of the two elements. In this case, fuzzy AND would be an appropriate combination operator, because at each location the combination would be controlled by the minimum of the fuzzy membership values. In other situations, fuzzy OR is more appropriate, where for example, the presence of any one of the pathfinder elements in abundance might he significant evidence for the presence of a mineral deposit, even though other pathfinder elements are not present in anomalous amounts.

Evidence maps can be combined together in a series of steps, as depicted in an inference network, Figure 9-7. Thus instead of combining all the maps in one operation, for example with the gamma operator, it may be more appropriate to link together some maps with, say the fuzzy OR to support an intermediate hypothesis, other maps with fuzzy AND to support another intermediate hypothesis, and finally to link both raw evidence and intermediate hypotheses (now in turn being used as evidence) with a fuzzy gamma operation. Many combinations are possible. The inference network becomes an important means of simulating the logical thought processes of an expert. In expert system terminology, the fuzzy membership functions are the "knowledge base" and the inference network and fuzzy combination rules are the "inference engine". Fuzzy logic is one of the tools used in expert systems where the uncertainty of evidence is important. Even quite complex inference networks can be implemented in a map modelling language. Fuzzy logic has also been applied to problems of pattern recognition in geology, see Griffiths (1987).



Landfill site selection output map: Suitability using fuzzy logic. The output is ranked according to membership of a fuzzy set, the set comprising areas that satisfy the proposition "*This location is suitable for a landfill*". The degree with which the proposition is satisfied is scored on a scale from 0 to 1, then classified to make a map.

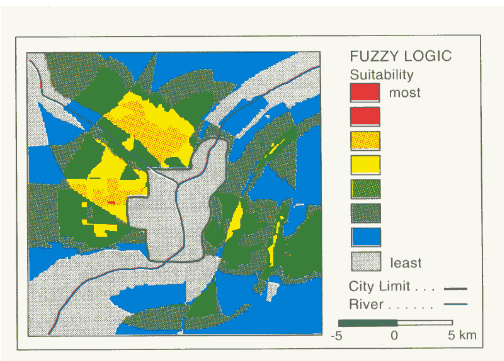


Fig. 9-2D

Output maps showing gold potential: Areas ranked according to fuzzy membership values. The fuzzy set comprises those locations that satisfy the proposition "*favourable for gold exploration*", and fuzzy membership ranges from 0 to 1. Most of the main known gold occurrences are predicted by the model, and several new prospective areas are suggested.

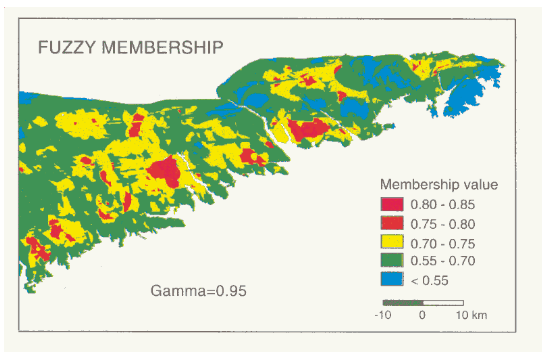


Fig. 9-4D